英

秋季募集（令和６年度実施）

東北大学大学院工学研究科
量子エネルギー工学専攻入学試験

試験問題冊子

数学A　MATHEMATICS A

令和６年８月２７日(火)　10:00 ～11:30

Tuesday, August 27, 2024　10:00 ～11:30

Notice

1. Do not open this examination booklet until instructed to do so.

2. An examination booklet, answer sheets, draft sheets are provided. Put your entrance examination ID-No. on each of the answer sheets and the draft sheets.

3. Answer all problems. Indicate the problem number on the answer sheets.

4. At the end of the examination, double-check your entrance examination ID-No. and the problem numbers on the answer sheets. Put your answer sheets in numerical order on your draft sheets, place them beside the test booklet, and wait for collection by an examiner. Do not leave your seat before instructed to do so by the examiner.

数 学 A MATHEMATICS A

1. Consider the following function,

$$y = f(x) = \frac{1 - \cos x}{x^2}.$$

Solve the following problems.

(1) Evaluate an approximation for $f(x)$ up to x^4 term by using the result of Taylor expansion of $\cos x$ around $x = 0$.

(2) Evaluate the limit of $y = f(x)$ when $x \to 0$.

(3) Draw the graph of $y = f(x)$ on the xy-plane for $0 < x \leq 2\pi$.

(4) Determine whether the improper integral, $\lim_{\varepsilon \to +0} \int_{\varepsilon}^{\infty} f(x)\, dx$, converges or not by dividing the interval of integration into $0 \leq x \leq 1$ and $x \geq 1$.

2. In the three-dimensional Cartesian coordinate system (x, y, z), a vector field A is given by

$$A = y\sqrt{x^2+y^2+z^2}\,\boldsymbol{i} - x\sqrt{x^2+y^2+z^2}\,\boldsymbol{j} + z\sqrt{x^2+y^2}\,\boldsymbol{k},$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are the fundamental vectors in the x, y, and z directions, respectively. In addition, a region D is given by

$$D = \{\,(x, y, z) \mid x^2+y^2+z^2 \le 2,\ x^2+y^2-z^2 \ge 0, z \ge 0\,\}.$$

Surfaces of the region D is defined as a curved surface S. Here the cylindrical and three-dimensional polar coordinate systems are expressed as (ρ, ϕ, z) and (r, θ, ϕ), respectively. Solve the following problems.

(1) Obtain $\nabla \cdot A$ and $\nabla \times A$ in the three-dimensional Cartesian coordinate system.

(2) Express x, y, and z in the cylindrical coordinate system.

(3) Draw $x^2+y^2-z^2=0$ $(0 \le z \le 1)$ in the three-dimensional Cartesian coordinate system. If necessary, use the results obtained in problem (2).

(4) Obtain the Jacobian for the change of variables from three-dimensional Cartesian coordinates to three-dimensional polar coordinates.

(5) Evaluate the surface integral $\int_S A \cdot \boldsymbol{n}\, dS$, where $\boldsymbol{n}$ is the outward unit normal vector of S.

数 学 A　MATHEMATICS A

3. The 2×2 matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A}=\begin{pmatrix} 7a+9 & 24a-12 \\ 24a-12 & 16-7a \end{pmatrix}.$$

Solve the following problems. Here, a is a positive constant and $a\neq\frac{1}{2}$.

(1) Show that the eigenvalues of $\boldsymbol{A}$ are $25a$ and $25(1-a)$ from the characteristic equation.

(2) Find two normalized eigenvectors of $\boldsymbol{A}$.

(3) Using the results obtained in problem (2), find the 2×2 diagonal matrix $\boldsymbol{D}$ and the 2×2 matrix $\boldsymbol{P}$ that satisfy $\boldsymbol{A}=\boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$. Here the diagonal elements of $\boldsymbol{P}$ are positive.

(4) Using the results obtained in problem (3), describe shapes of the following function,

$$f(x,y)=(7a+9)x^2-2(24a-12)xy+(16-7a)y^2=1.$$