2. Consider a two-dimensional steady potential flow of an inviscid incompressible fluid. Answer the following questions.

(1) The flow around a vortex filament located at the origin is given by the following complex velocity potential,

$$W(z) = i\,k \ln z \ ,$$

where k is a positive real number, i is the imaginary unit, ln is the natural logarithm, z is a complex variable which is expressed by $z = re^{i\theta}$, and r and θ are radial and circumferential coordinates, respectively.

a) Obtain the velocity potential $\Phi(r, \theta)$ and the stream function $\Psi(r, \theta)$.

b) Obtain the counterclockwise circulation Γ along the circle of radius r centered at the origin.

c) Answer whether the circulating flow around the vortex filament is a free vortex or a forced vortex.

(2) As shown in Fig. 3, three vortex filaments A, B and C with circulation Γ_A, Γ_B and Γ_C, respectively, are aligned at regular intervals of h. Assuming an infinite domain of fluid where the fluid velocity approaches to zero at infinity, a vortex filament moves with the velocity induced by the other vortex filaments. Obtain the relationship among Γ_A, Γ_B and Γ_C in the case when all the vortex filaments stand still in the flow field. Here, Γ_A is assumed to be nonzero.

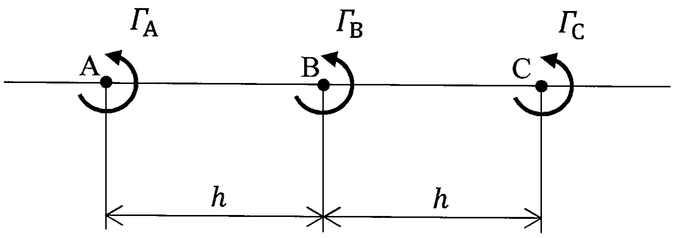

Fig. 3

材 料 力 学　STRENGTH OF MATERIALS

1. As shown in Fig. 1, a two-dimensional truss composed of four uniform bars AF, BF, CF, and DF is pin-connected at their ends on a horizontal rigid ceiling and point F. The distance from point F to the rigid ceiling is h. The angle of the bars BF and CF from the vertical direction is θ_1 and that of AF and DF is θ_2. Assume $0 < \theta_1 < \theta_2 < \pi/2$. Young's modulus and the cross-sectional area of the four bars are E and S, respectively. A vertical downward load P is applied at point F. Neglect the weight of the bars. Answer the following questions.

(1) Denote the axial forces in AF, BF, CF, and DF by R_{AF}, R_{BF}, R_{CF}, and R_{DF}, respectively. Derive the equations of force equilibrium in both horizontal and vertical directions for the truss.

(2) Show the axial strain of each bar ε_{AF}, ε_{BF}, ε_{CF}, and ε_{DF} in terms of R_{AF}, R_{BF}, R_{CF}, and R_{DF}.

(3) Indicate the relationship between the elongation of each bar Δl_{AF}, Δl_{BF}, Δl_{CF}, and Δl_{DF}, and the vertical displacement δ at point F.

(4) Determine the axial forces R_{AF}, R_{BF}, R_{CF}, and R_{DF}.

(5) Find the vertical displacement δ at point F.

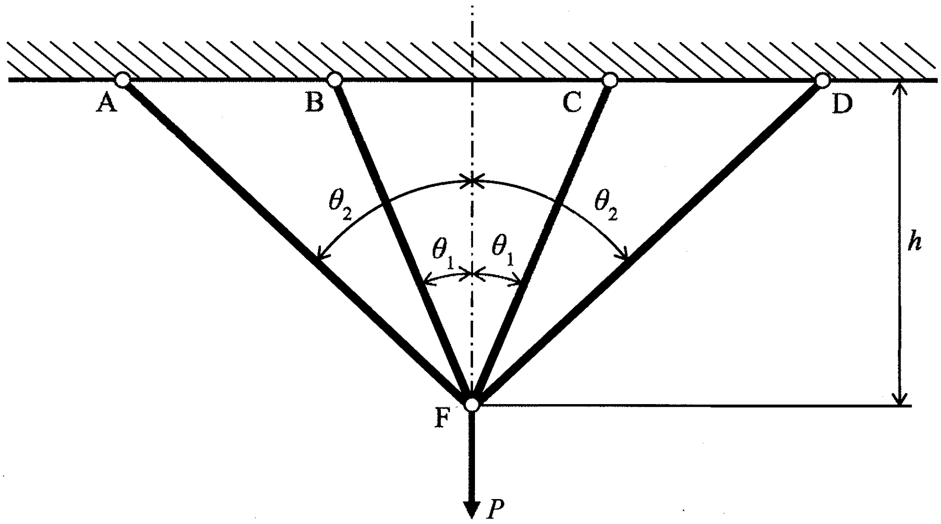

Fig. 1

2. Consider an L-shaped frame ABC which consists of beams AB and BC. The frame is fixed at point A. The length of the beams AB and BC is $2L$, and flexural rigidity EI of the beams is constant. Neglect both the weight of the beams and the elongation of the neutral axis of the beams. Answer the following questions.

(1) A bending moment M_B is applied at point B, as shown in Fig. 2(a). Determine the deflection angle at point B.
(2) A load W is applied at the middle point D of BC vertically in the downward direction, as shown in Fig. 2(b). Determine the deflection of the frame at right end C.
(3) The L-shaped frame ABC is simply supported at right end C, as shown in Fig. 2(c). Determine the reaction at right end C, when a load W is applied at point D vertically in the downward direction.

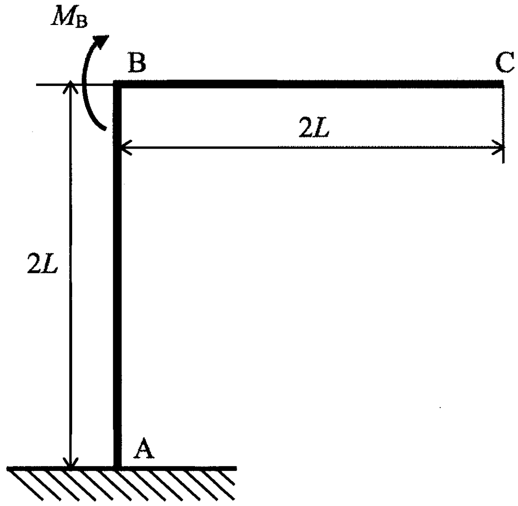

Fig. 2(a)

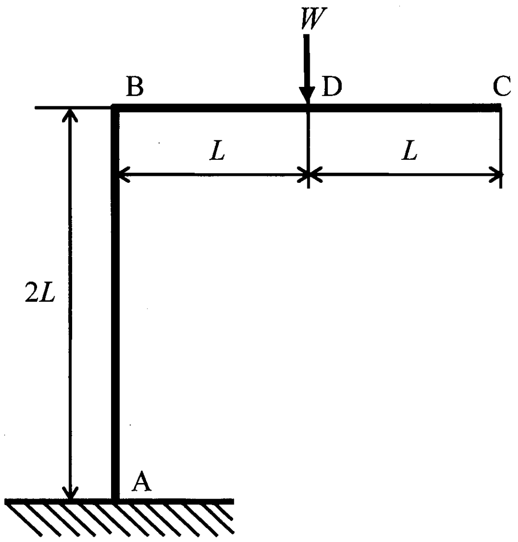

Fig. 2(b)

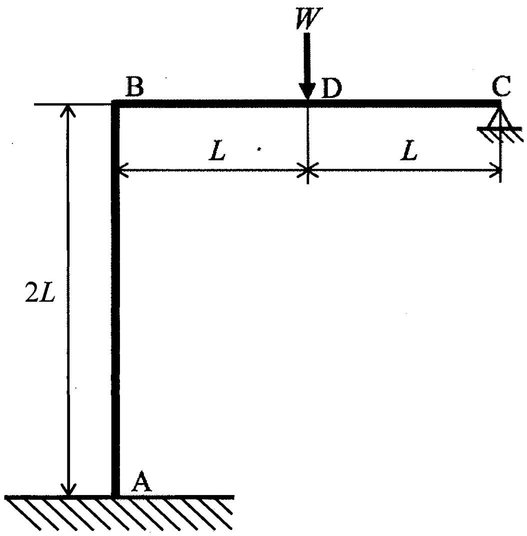

Fig. 2(c)

1. Consider a system consisting of a mass m, two springs with spring constants k_1 and k_2, a dashpot with damping coefficient c, and a beam with length L, Young's modulus E and moment of inertia of area I, as shown in Fig. 1. Assume that the masses of the springs, the dashpot and the beam are negligible, and that the mass m vibrates only in the horizontal direction with a sufficiently small displacement. The displacement of the mass m from the equilibrium position is denoted by $x(t)$, where t is time. Answer the following questions.

(1) Express the equivalent spring constant k_3 of the beam by using L, E and I.

(2) Find the equivalent spring constant K of the system.

(3) Obtain the critical damping coefficient of the system.

(4) Find the damped natural period T of the system, when the system is underdamped.

(5) Obtain the ratio of $x(t)$ at two different times separated by the period T under the condition of question (4) when $x(t) \neq 0$.

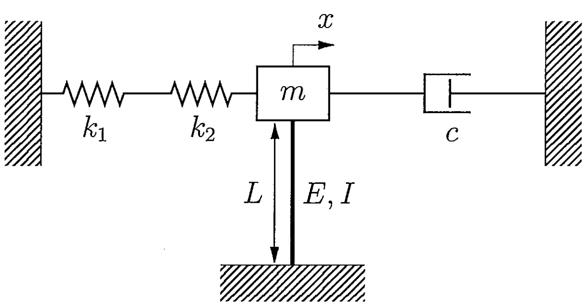

Fig. 1

2. Consider a system consisting of a uniform rotating disk with mass m_1, radius r and moment of inertia J, two springs with spring constants k_1 and k_2, and a trailer with mass m_2, as shown in Fig. 2. The right end of the spring with spring constant k_1 is connected to the center axis O of the disk through a frictionless bearing, and the left end is fixed to the trailer. The right end of the spring with spring constant k_2 is connected to the trailer, and the left end is fixed to the wall. The disk rotates on the trailer without slipping and the trailer vibrates only in the horizontal direction. The angular displacement of the disk and the displacement of the trailer from the equilibrium positions are denoted by θ and x, respectively. Assumimg that the masses of the springs are negligible, answer the following questions.

(1) Obtain the kinetic energy T of the system.

(2) Obtain the potential energy U of the system.

(3) Derive the equations of motion of the system.

(4) When $m_1 = 2m$, $m_2 = m$, and $k_1 = k_2 = k$, find the natural angular frequencies of the system using $J = mr^2$.

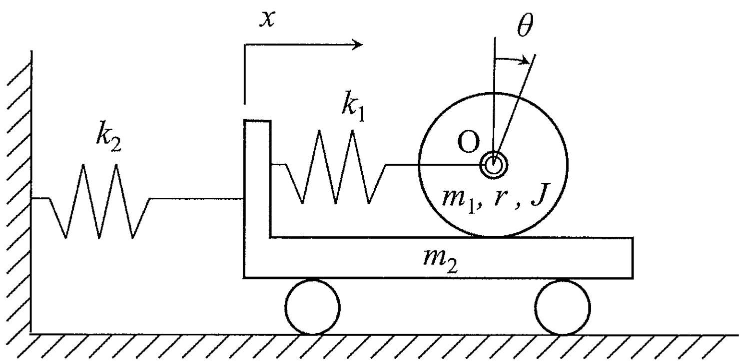

Fig. 2

制 御 工 学 CONTROL ENGINEERING

1. Solve the following problems. s is the Laplace operator and t is time.

 (1) Derive the unit step response of the system shown in Fig. 1 and draw its outline.

 (2) Obtain the transfer function $G(s)$ from $U(s)$ to $Y(s)$ of the system shown in Fig. 1. Obtain also the poles and zeros of the transfer function.

 (3) Consider a feedback control system shown in Fig. 2. The transfer function of the controlled system $P(s)$ is given by

$$P(s) = \frac{4}{(s+1)(s+4)}.$$

When a unit ramp function $r(t) = t$ is applied as a reference input, the system has a steady-state velocity error e_v $(0 < e_v < \infty)$. In this case, select a controller $C(s)$ from (i) – (iii) and explain the reason. Note K_1, K_2, and K_3 are positive constants.

$$\text{(i)} \quad C(s) = K_1, \quad \text{(ii)} \quad C(s) = \frac{K_1 s + K_2}{s}, \quad \text{(iii)} \quad C(s) = \frac{K_1 s^2 + K_2 s + K_3}{s^2}.$$

 (4) Consider the feedback control system in problem (3). Find the conditions so that the system is stable. Find also the conditions so that the steady-state velocity error e_v of the system satisfies $e_v \leq 0.1$.

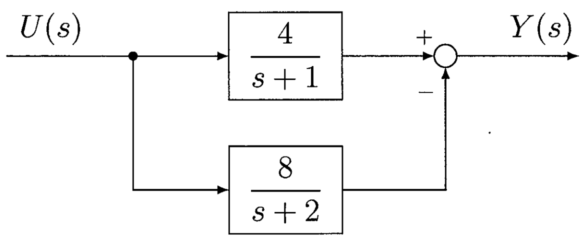

Fig. 1

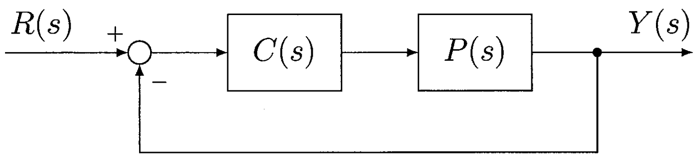

Fig. 2

2. Consider the following controllable linear time-invariant system

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u},$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x},$$

where $\boldsymbol{x}$ is a state vector, $\boldsymbol{u}$ is input, and $\boldsymbol{y}$ is output.

It is known that the optimal control input to minimize a criterion function

$$J = \frac{1}{2}\int_0^\infty (\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u})dt$$

is given by $\boldsymbol{u} = -\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x}$, where $\boldsymbol{P}$ is the positive definite solution of the following Ricatti equation,

$$\boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{P} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P} + \boldsymbol{Q} = \boldsymbol{O},$$

where $\boldsymbol{Q}$ and $\boldsymbol{R}$ denote the weight matrices, and $\boldsymbol{O}$ denotes the zero matrix. Solve the following problems.

(1) Consider the electric circuit shown in Fig. 3 in which the inductance, the resistance, and the capacitance are given by l, r, and c, respectively. The input and output voltages of the circuit are given by e_i and e_o, respectively. Let i be the current flowing through the coil and v be the voltage applied to the capacitor. Derive the state equation and the output equation of the system in which the state vector, the input, and the output are defined as $\boldsymbol{x} = (i, v)^T$, $u = e_i$, and $y = e_o$, respectively.

(2) Obtain the matrix $\boldsymbol{P} = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}$ and the optimal control input u, where the circuit constants are given as $l = r = c = 1$, and the weights are given as $R = 1$ and $\boldsymbol{Q} = \begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix}$ in problem (1).

(3) Explain how the initial value response of the system is influenced, when $\boldsymbol{Q}$ is not changed while R is increased in problem (2).

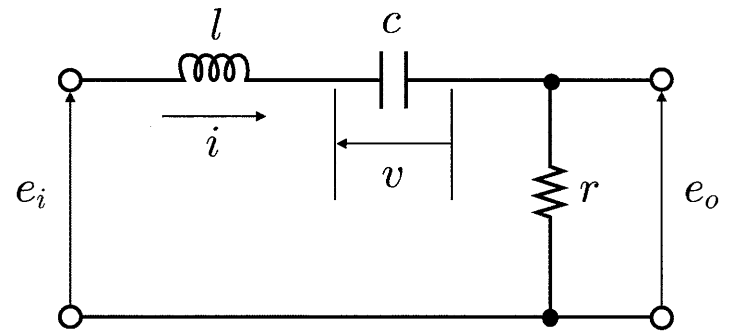

Fig. 3

材 料 物 性 学　MATERIALS SCIENCE

1. Fig.1 shows a portion of Fe-C phase diagram. Answer the following questions.

(1) Write the names of all phases existing in regions **[1]**, **[2]**, **[3]** and **[4]** of Fig. 1, respectively.

(2) Consider that a steel **X** (carbon content 0.316 mass%) and a steel **Y** (carbon content 1.200 mass%) are slowly cooled down from the region **[1]**. Draw schematics of the metallographic structure corresponding to points **h**, **i**, **j**, **m** and **n**, indicating the distribution of each phase. Here, refer to the schematic of metallographic structure shown in Fig. 1.

(3) Obtain the ratio of ferrite at point **R** in the region **[2]** just above the line of A_1 transformation. For the calculation, assume that the carbon content at points **P** and **Q** are 0.020 and 0.760 mass%, respectively.

(4) Consider steels **X** and **Y** which are slowly cooled down from the region **[1]** to room temperature. Describe the differences in the mechanical properties between these steels and explain the reasons of the differences.

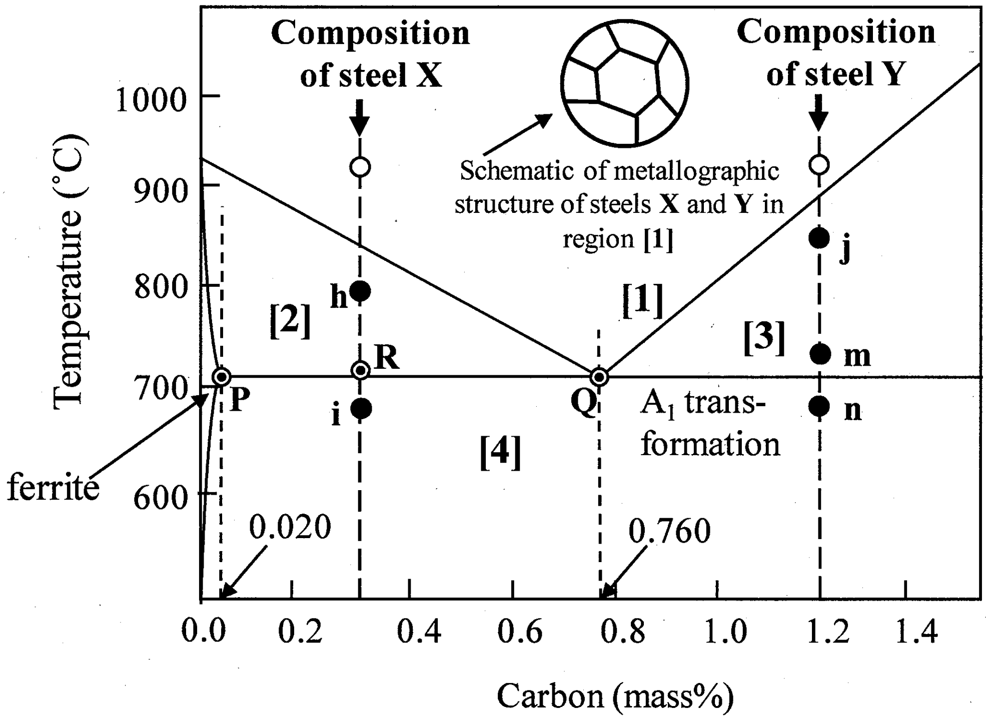

Fig. 1

2. Answer the following questions about fatigue of metals in an inert environment at ambient temperature. Here, S is stress amplitude and N is a number of loading cycles to failure.

(1) Draw an outline of S - N curve for a mild steel under sinusoidal cyclic loading with a constant S.

(2) Explain the linear cumulative damage rule (Miner's law) about N under sinusoidal cyclic loading with a variable S.

(3) Explain the mechanism of fatigue of metals.

(4) It is known that fatigue crack growth rate (defined as crack length increment per loading cycle) is governed by stress intensity factor range, ΔK. ΔK is defined as the following equation by the maximum stress intensity factor, K_{max}, and the minimum stress intensity factor, K_{min}, in the loading cycle.

$$\Delta K = K_{max} - K_{min} \quad (\text{when } K_{min} \geqq 0)$$

Draw an outline of a diagram showing the relationship between ΔK and crack growth rate, and explain Paris's law about fatigue crack growth rate.

電 磁 気 学　ELECTROMAGNETICS

1. Consider a disk C with radius R_0 on the plane perpendicular to the z-axis as shown in Fig. 1. The thickness of the disk C is assumed to be 0. On the disk C, an electric charge is uniformly distributed with a surface charge density σ (> 0). Answer the following questions. Use ε_0 for the permittivity.

   (1) Consider a ring (inner radius R, outer radius $R + dR$, and $dR << R$) on the disk C as shown in Fig. 1. Find the magnitude and the direction of the electric field $\boldsymbol{E}$ at the point P(0, 0, z_p) on the z-axis generated by the charge on the ring.

   (2) Using the result of question (1), find the magnitude and the direction of the electric field $\boldsymbol{E}$ at the point P generated by the charge on the disk C.

   (3) Using the result of question (2), find the magnitude and the direction of the electric field $\boldsymbol{E}$ at the point P when the radius R_0 is infinitely large.

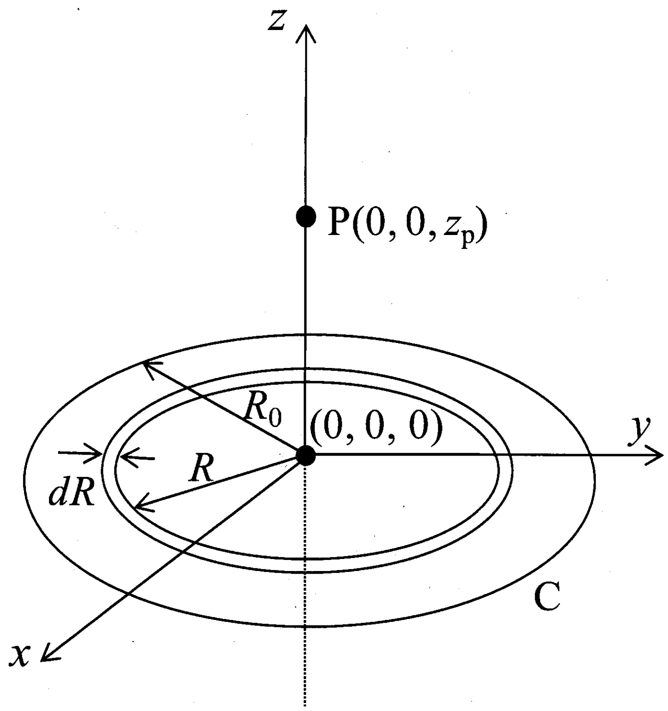

Fig. 1

2. As shown in Fig. 2, magnetic field $\boldsymbol{H}(t)=H_0\boldsymbol{k}\sin(\omega t)$ is uniformly applied in the z-direction to a square loop with a side length of a, where H_0 is a constant, $\boldsymbol{k}$ is a unit vector in the z-direction, ω is an angular velocity, and t is time. The square loop can rotate on the x-axis with an angular velocity of ω_L. θ is an angle between the square loop and the xy-plane, and the resistance of the square loop is R. Neglect the magnetic field generated by the induced current. Use μ_0 for the permeability.

(1) When $\omega_L \ll \omega$, answer the following questions.

a) Find electromotive force induced in the square loop and the current flowing in the square loop as a function of the angle θ.

b) Find the maximum Joule's heat generated in the square loop and the corresponding angle θ.

c) When the angle θ is $\pi/6$, find the torque exerted on the square loop.

(2) When $\omega_L = \omega$ and the rotation angle is expressed as $\theta=\omega t$, answer the following questions.

a) Find the maximum current flowing in the square loop and the rotation angle at that time.

b) Find the total charge flowing in the square loop while the angle θ changes from 0 to $\pi/6$.

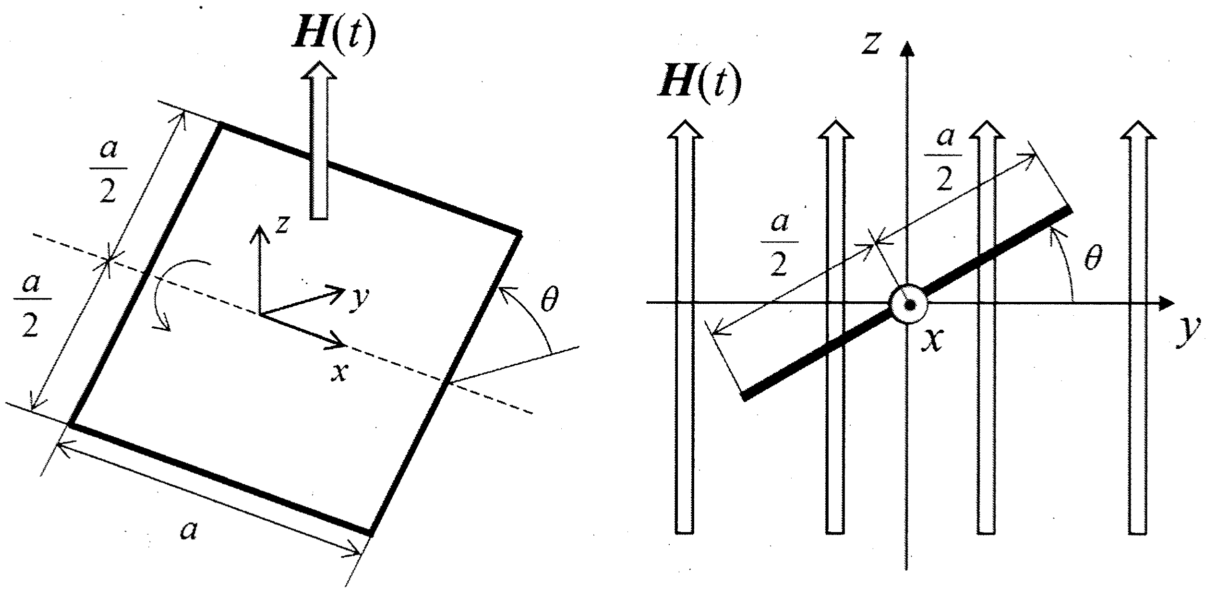

Fig. 2

量 子 力 学　QUANTUM MECHANICS

1. Answer the following questions using elemental charge e, light velocity in vacuum c, Boltzmann's constant k, Planck's constant h and $\hbar = h/(2\pi)$. Here, assume that the effect of relativistic kinematics in particle motion can be ignored.

   (1) When a neutron with mass m_N is in equilibrium at an absolute temperature T, express the wavelength λ_N of the neutron using k, m_N, h and T.

   (2) Consider a photoelectron emitted from the metal surface which is irradiated by a photon with wavelength λ. When the photoelectron perpendicularly enters the magnetic field of magnetic flux density $\boldsymbol{B}$, and moves in a circular orbit with radius R, express the work function ϕ of the metal using λ, e, c, R, $\boldsymbol{B}$, h and electron mass m.

   (3) When a free particle is confined in a two-dimensional region of $-a \le x \le a$ and $-b \le y \le b$, find the minimum kinetic energy of the particle on the basis of the uncertainty relation between position and momentum $\Delta q \Delta P \le \hbar/2$, where q is position and P is momentum.

   (4) The radial wave functions $R_{nl}(r)$ for the $2s$ and $2p$ orbitals for an electron of a hydrogen atom are given by

$$R_{20}(r) = (2a_0)^{-3/2}(2 - r/a_0)e^{-r/2a_0},$$
$$R_{21}(r) = \left(1/\sqrt{3}\right)(2a_0)^{-3/2}(r/a_0)e^{-r/2a_0},$$

   respectively, where n is the radial quantum number, l is the orbital angular momentum quantum number, r is the distance from the nucleus, and a_0 is the Bohr radius. Show that the $2s$-orbital electron exists on the average further from the nucleus than the $2p$-orbital electron does.

量 子 力 学　QUANTUM MECHANICS

2. The one-dimensional time-dependent Schrödinger equation is given by

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}+V(x)\psi(x,t)=i\hbar\frac{\partial\psi(x,t)}{\partial t} ,$$

where m is mass of a particle, $\psi(x,t)$ is a wave function describing the state of the particle, the $V(x)$ is a real potential energy, $\hbar$ is denoted by $h/(2\pi)$, and h is Planck's constant. Assuming that the energy of this system is E, answer the following questions.

(1) Putting $\psi(x,t)=X(x)T(t)$, obtain the ordinary differential equations in terms of $X(x)$ and $T(t)$, respectively.

(2) Give the solution of $T(t)$ in question (1).

(3) When $V(x)=x^2/2$, show the limiting values of $X(x)$ at $x=\pm\infty$, and explain its physical meaning.

(4) When $V(x)$ is an even function, show that $X(x)=X(-x)$ or $X(x)=-X(-x)$.