平成 28 年度 秋季募集
(平成 28 年 10 月・平成 29 年 4 月入学)
東北大学大学院機械・知能系入学試験
試験問題冊子

数学 A  MATHEMATICS A

平成 28 年 8 月 29 日(月)
Monday, August 29, 2016  9:30 - 11:00

Notice

1. Do not open this test booklet until instructed to do so.

2. A test booklet, answer sheets, draft sheets, and a selected-problems form are provided. Put your examinee number on each of the answer sheets, the draft sheets, and the form.

3. Select three of the four problems and answer them. Indicate your selection on the selected-problems form. Use two answer sheets for each problem.

4. At the end of the examination, double-check your examinee number and the problem numbers on the answer sheets. Put your answer sheets in numerical order on top of the other sheets, place them beside the test booklet, and wait for collection by an examiner. Do not leave your seat before instructed to do so by the examiner.
1. Let

\[ P(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \]

where \( \lambda \) is a positive real number and \( k \) is an integer greater than or equal to 0. Solve the following problems.

(1) Sketch the graph of \( P(k) \) with \( k \) as the horizontal axis and \( 0 \leq k \leq 8 \) when \( \lambda = 2 \).

(2) Show the Taylor series of \( e^x \) about \( x = 0 \).

(3) Evaluate the following infinite series.
   a) \( S_1 = \sum_{k=0}^{\infty} P(k) \)
   b) \( S_2 = \sum_{k=0}^{\infty} kP(k) \)
   c) \( S_3 = \sum_{k=0}^{\infty} (k - S_2)^2 P(k) \)

(4) Show that

\[ \sum_{k=0}^{n} P(k) = 1 - \frac{1}{n!} \int_{0}^{\lambda} t^n e^{-t} dt, \]

where \( n \) is a positive integer.
2. Find the general solutions of the following ordinary differential equations.

(1) \((x^2 + y^2) \frac{dy}{dx} - xy = 0\)

(2) \(\frac{dy}{dx} + 2y \tan x = \sin x\)

(3) \(\frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + y = e^{-x} \log x\)
3. The matrices $A$, $B$, and $C$ are given by

$$A = \begin{pmatrix} 3 & 1 & 3 \\ 1 & 2 & -1 \\ 3 & -1 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -1 & 1 \\ -2 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix}, \quad C = AB.$$ 

Solve the following problems.

(1) Obtain $C$.

(2) Find the eigenvalues and eigenvectors of $C$.

(3) Find a regular matrix $P$ and its inverse matrix $P^{-1}$ such that $P^{-1}CP$ is a diagonal matrix.

(4) Obtain $C^n$, where $n$ is a positive integer.
4. The surface $S$ is given by

$$r(r, \theta) = (r \cos \theta) \hat{i} + (r \sin \theta) \hat{j} + \theta \hat{k} \quad \left(0 \leq r \leq a, \ 0 \leq \theta \leq \frac{\pi}{2}\right),$$

where $\hat{i}$, $\hat{j}$, and $\hat{k}$ are the fundamental vectors in a Cartesian coordinate system $(x, y, z)$ and $a$ is a positive constant. Solve the following problems.

(1) Obtain the unit normal vector $\mathbf{n}$ of $S$.

(2) Evaluate

$$\int_S \mathbf{A} \cdot \mathbf{n} dS,$$

when the vector field $\mathbf{A}$ is given by

$$\mathbf{A} = x \hat{i} + y \hat{j} + z \hat{k}.$$ 

(3) Evaluate the length of the curve that is obtained on $S$ with $r = a$.

(4) Evaluate the area of $S$. 

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試験問題冊子

数学 B \hspace{1cm} MATHEMATICS B

平成 28 年 8 月 29 日(月)
Monday, August 29, 2016 \hspace{1cm} 13:30 - 15:00

Notice

1. Do not open this test booklet until instructed to do so.

2. A test booklet, answer sheets, draft sheets, and a selected-problems form are provided. Put your examinee number on each of the answer sheets, the draft sheets, and the form.

3. Select two of the three problems, and answer them. Indicate your selection on the selected-problems form. Use two answer sheets for each problem.

4. At the end of the examination, double-check your examinee number and the problem numbers on the answer sheets. Put your answer sheets in numerical order on top of the other sheets, place them beside the test booklet, and wait for collection by an examiner. Do not leave your seat before instructed to do so by the examiner.
1. The Fourier cosine transform $F_c(\omega)$ of an even function $f(t)$ is defined by

$$F_c(\omega) = \int_0^\infty f(t) \cos \omega t \, dt.$$ 

Using $f(t)$, the function $g(t)$ with period $T$ is defined by

$$g(t) = \sum_{n=-\infty}^{\infty} f(t - nT),$$

where $n$ is an integer. Solve the following problems.

(1) Obtain the Fourier cosine transform of $e^{-|t|}$.

(2) Show that $\int_{-T/2}^{T/2} g(t) \, dt = \int_{-\infty}^{\infty} f(t) \, dt$.

(3) On the interval $-T/2 \leq t \leq T/2$, the Fourier series of $g(t)$ is given by

$$g(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos \left(\frac{2n\pi t}{T}\right) + b_n \sin \left(\frac{2n\pi t}{T}\right) \right\}.$$ 

Express the Fourier coefficients $a_0$, $a_n$, and $b_n$ using the Fourier cosine transform $F_c(\omega)$ of the even function $f(t)$.

(4) Show that $\sum_{n=-\infty}^{\infty} e^{-|n|} = \sum_{n=-\infty}^{\infty} \frac{2}{(2n\pi)^2 + 1}$. 

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2. The Laplace transform of a function $f(t)$ is defined by

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t) e^{-st} \, dt.$$ 

Solve the following problems.

(1) When $f(t) = \sqrt{t}$, express the Laplace transform of $\frac{1}{\sqrt{t}}$ using $\mathcal{L}[f(t)]$.

(2) Obtain the Laplace transform of $\sqrt{t}$.

(Hint: $\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}$.)

(3) When $n$ is a non-negative integer, obtain the Laplace transform of

$$\frac{t^{n+\frac{1}{2}}}{(n + \frac{1}{2}) (n - \frac{1}{2}) (n - \frac{3}{2}) \cdots \frac{1}{2} \cdot \frac{1}{2}}.$$ 

(Hint: $\lim_{t \to \infty} t^{n+\frac{1}{2}} e^{-st} = 0$.)
3. The function \( u(x, y, t) \) satisfies the partial differential equation

\[
\frac{\partial u}{\partial t} = \alpha \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (0 < x < L, \ 0 < y < L, \ 0 < t)
\]

with the boundary conditions

\[
u(0, y, t) = u(L, y, t) = u(x, 0, t) = u(x, L, t) = 0.
\]

When \( \alpha \) and \( L \) are positive constants, solve the following problems.

(1) Let \( u(x, y, t) = f(x)g(y)h(t) \).

a) Show that the differential equations for \( f(x) \) and \( g(y) \) are given by

\[
\frac{d^2 f(x)}{dx^2} = -\lambda f(x), \quad \frac{d^2 g(y)}{dy^2} = -\eta g(y),
\]

respectively, where \( \lambda \) and \( \eta \) are positive constants.

b) Obtain the differential equation for \( h(t) \).

(2) Show that \( u(x, y, t) \) is given by

\[
u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \left( \frac{m\pi x}{L} \right) \sin \left( \frac{n\pi y}{L} \right) e^{-\frac{m^2 \pi^2 + n^2 \pi^2}{L^2} at},
\]

where \( A_{mn} \)'s are constants.

(3) Obtain \( A_{mn} \) in problem (2), when \( u(x, y, t) \) satisfies the boundary condition

\[
u(x, y, 0) = x(x-L) \sin \left( \frac{2\pi y}{L} \right).
\]
### Specialized Subjects

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<tr>
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<tr>
<td>熱力学</td>
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<td>流体力学</td>
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**平成 28 年 8 月 30 日 (火)**
Tuesday, August 30, 2016  9:00 — 12:00

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**Notice**

1. Do not open this test booklet until instructed to do so.

2. A test booklet, answer sheets, draft sheets, and two selected-subjects forms are provided. Put your examinee number on each of the answer sheets, the draft sheets, and the form.

3. Select two subjects from the eight subjects in the booklet and answer all problems in each subject. Indicate your selection on the selected-subjects form. Use one set of two answer sheets for each subject, and use one sheet per problem.

4. At the end of the examination, double-check your examinee number and the problem numbers on the answer sheets. Put your answer sheets in numerical order on top of the other sheets, place them beside the test booklet, and wait for collection by an examiner. Do not leave your seat before instructed to do so by the examiner.
1. Answer the following questions.

(1) Explain the following technical terms briefly.
   a) Thermodynamic equilibrium
   b) Extensive quantity (extensive property)
   c) Superheated vapor
   d) Helmholtz free energy

(2) Pressure, temperature, specific volume, specific internal energy, specific entropy and specific heat at constant volume are expressed as $p$, $T$, $v$, $u$, $s$ and $c_v$, respectively. Answer the following questions. Use the following Maxwell thermodynamic relations, if necessary.

$$
\left( \frac{\partial p}{\partial T} \right)_v = \left( \frac{\partial s}{\partial v} \right)_T
$$

$$
\left( \frac{\partial v}{\partial T} \right)_p = - \left( \frac{\partial s}{\partial p} \right)_T
$$

a) Derive the following equation.

$$
c_v = T \left( \frac{\partial s}{\partial T} \right)_v
$$

b) Show the total differential of $s$ when $s$ is a function of $T$ and $v$.

c) Derive the following equation.

$$
du = c_v dT + \left[ T \left( \frac{\partial p}{\partial T} \right)_v - p \right] dv
$$

d) Show that an ideal gas satisfies the following equations.

$$
\left( \frac{\partial u}{\partial v} \right)_T = 0
$$

$$
\left( \frac{\partial u}{\partial p} \right)_T = 0
$$
2. Consider a cycle using an ideal gas in a closed system. The cycle consists of three quasi-static processes, which are an isochoric heating process of state $1 \rightarrow 2$, an adiabatic expansion process of state $2 \rightarrow 3$ and an isobaric cooling process of state $3 \rightarrow 1$. The pressure and specific volume at state 1 are $p_1$ and $v_1$, respectively. The pressure at state 2 is $p_2$. The specific heat ratio and the gas constant of the ideal gas are $\kappa$ and $R$, respectively. Answer the following questions. Use the symbols $p_1$, $v_1$, $p_2$, $\kappa$ and $R$, if necessary.

(1) Draw the pressure - specific volume ($p$-$v$) diagram and the temperature - specific entropy ($T$-$s$) diagram of the cycle.

(2) Show the temperatures at the states 1, 2 and 3.

(3) Show the heat added to the cycle during the processes, $1 \rightarrow 2$, $2 \rightarrow 3$ and $3 \rightarrow 1$, respectively. The sign of heat added to the system is positive.

(4) Show the thermal efficiency of the cycle.
1. Consider a two-dimensional steady potential flow of an inviscid incompressible fluid as shown in Fig. 1. Here, a source of strength \( m \) (\( m \): positive real number) is located at the origin O in a uniform flow in the \( x \)-direction with the velocity of \( U \). The fluids of the source and the uniform flow are the same. A blunt-body-shaped streamline passing the stagnation point S on the \( x \)-axis is formed. The complex velocity potential of the flow is given by \( W(z) = Uz + m \log z \). The blunt-body-shaped streamline has a constant width of \( b \) at infinity downstream. The distance between the stagnation point S and the origin O is \( a \). Here, \( \log \) is the natural logarithm and \( z \) is the complex variable in the polar form given by \( z = re^{i\theta} \) where \( r \) and \( \theta \) are radial and circumferential coordinates, respectively, and \( i = \sqrt{-1} \). Answer the following questions.

(1) The complex velocity potential \( W(z) \) is defined as \( W(z) = \phi + i\psi \), where \( \phi \) is velocity potential and \( \psi \) is stream function. Express the functions \( \phi \) and \( \psi \) using \( U, m, r \) and \( \theta \).

(2) Express the radial velocity \( u_r \) using \( U, m, r \) and \( \theta \), and express the circumferential velocity \( u_\theta \) using \( U \) and \( \theta \).

(3) Express the distance \( a \) between the stagnation point S and the origin O using \( U \) and \( m \).

(4) Express the width \( b \) of the blunt body at infinity downstream using \( U \) and \( m \).

(5) The blunt-body-shaped streamline is given by \( r = f(\theta) \). Express \( f(\theta) \) using \( U, m \) and \( \theta \).

![Fig. 1](image-url)
2. An incompressible Newtonian fluid is flowing through a smooth straight horizontal circular pipe toward the positive \( x \)-direction under a fully developed laminar condition, as shown in Fig. 2. The density \( \rho \) and viscosity \( \mu \) are constant. The central axis of the circular pipe is the \( x \)-axis and the pipe radius is \( R \). The velocity components perpendicular to the \( x \)-axis are zero. Consider a circular cylinder of radius \( r \) and length \( l \), which is concentric with the pipe, as a control volume. The uniform pressures \( p_1 \) and \( p_2 \) act on the left and right faces of the control volume, respectively. Furthermore, the uniform shear stress \( \tau \) acts on the circumferential area. It is assumed that the gravitational force can be neglected. Answer the following questions.

(1) Express the shear stress \( \tau (r) \) using \( p_1 \), \( p_2 \), \( r \) and \( l \) in accordance with the balance equation of forces.

(2) Express the velocity distribution \( u (r) \) using \( U \), \( R \) and \( r \) in accordance with Newton’s law of viscosity, considering the no-slip condition at the pipe wall. Here, \( U \) denotes the maximum velocity.

(3) Express the Reynolds number \( Re \), defined by the mean velocity and the pipe diameter \( 2R \), using \( \rho \), \( \mu \), \( U \) and \( R \).

(4) Express the momentum in the \( x \)-direction transported per unit time by the fluid flowing through the cross section perpendicular to the \( x \)-axis using \( \rho \), \( U \) and \( R \).

Fig. 2
1. A circular composite shaft of length $L$ consists of a hollow shaft ① and a solid shaft ② as shown in Fig. 1. The shaft ① has outer and inner diameters of $d_1$ and $d_2$, respectively, and its modulus of rigidity in shear is $G_1$. The shaft ② has diameter $d_2$, and its modulus of rigidity in shear is $G_2$. These shafts are firmly bonded each other. The composite shaft is fixed to a rigid wall at its left end and twisted by a moment $M_t$ at the right end. Answer the following questions.

(1) The polar moment of inertia of area of the shaft ① is $I_{p1}$ and that of the shaft ② is $I_{p2}$. Express $I_{p1}$ and $I_{p2}$ using $d_1$ and $d_2$.

(2) Obtain the angle of twist at the right end of the composite shaft in the case of $G_1 = G_2$.

(3) Obtain the angle of twist at the right end of the composite shaft in the case of $G_1 \neq G_2$.

(4) Obtain the ratio $G_1/G_2$ when the maximum shearing stress in the shaft ① equals that in the shaft ② in question (3).

Fig. 1
2. A straight beam AB of length $2L$ and a straight beam CD of length $L$ are horizontally fixed to rigid walls at one end and contacted each other at the other end as shown in Fig. 2. Both beams have a circular cross-section of diameter $d$, and their Young’s modulus is $E$. A concentrated load $W$ is applied vertically downward at point C. Neglect the weight of the beams and assume that $L$ is much larger than $d$. Answer the following questions.

(1) Determine the ratio of the load $W_{AB}/W_{CD}$, where $W_{AB}$ is the load which is applied to beam AB and $W_{CD}$ is the load which is applied to beam CD, respectively.

(2) Determine the deflection of beam AB at point A.

(3) Determine the magnitude of the maximum bending stress (tensile stress) in the beams and indicate its position.

Fig. 2
1. Consider a system consisting of a mass $m$, two springs with spring constants $k_1$ and $k_2$, and two dashpots with damping coefficients $c_1$ and $c_2$, as shown in Fig. 1. The mass $m$ is supported by wheels 1 and 2 through the springs and the dashpots, and vibrates only in the vertical direction. The wheels 1 and 2 keep contact with the sinusoidal rails 1 and 2 with amplitudes $a_1$ and $a_2$, and wavelengths $\ell_1$ and $\ell_2$. The horizontal velocity of the mass $m$ is $v$, and the two wheels pass over the highest positions A and B of the two rails at time $t = 0$. Here, the vertical displacement of the mass $m$ from the equilibrium position of the system $x$ is defined when two wheels are at the highest positions of the rails. Assume that the two rails are in the same plane, and do not intersect each other. The two wheels are sufficiently small, and the masses of the wheels, the springs, and the dashpots are negligible. When the system is in a steady-state, answer the following questions.

(1) Express the vertical displacement of the wheel 1 from the position A of the rail 1 as a function of $t$.

(2) Derive the equation of motion of the system.

(3) When $k_1 = k_2 = k$, $c_1 = c_2 = c$, $a_1 = a_2 = a$, and $\ell_1 = \ell_2 = \ell$, determine the velocity $v$ at which the amplitude of $x$ reaches the maximum, assuming $c < \sqrt{2mk}$.

Fig. 1
2. Consider a system consisting of a uniform disk with mass $m_1$ and radius $r$, two springs with spring constants $k_1$ and $k_2$, and a mass $m_2$, as shown in Fig. 2. The left end of the spring with spring constant $k_1$ is fixed to the wall and its right end is connected to the center of the disk $O$ through a frictionless bearing. The left end of the spring with spring constant $k_2$ is connected to the center of the disk $O$ through the bearing and its right end is connected to the mass $m_2$. The disk rolls on the floor without slipping, and the mass $m_2$ vibrates only in the horizontal direction with no friction. The angular displacement of the disk and the displacement of the mass $m_2$ from the equilibrium positions of the system are denoted by $\theta$ and $x$, respectively. Assuming that the masses of the springs are negligible, answer the following questions.

(1) Obtain the mass moment of inertia $J$ of the disk.

(2) Derive the kinetic energy and the potential energy of the system.

(3) Derive the equations of motion of the system.

(4) When $m_1 = \frac{8}{3}m_2$ and $k_1 = 3k_2$, express the natural angular frequencies of the system using $m_2$ and $k_2$.

(5) Find $\frac{x}{r\theta}$ at each natural angular frequency obtained in question (4).
1. Solve the following problems, where \( s \) is the Laplace operator, \( j \) is imaginary unit, \( \omega \) is angular frequency, and \( t \) is time.

(1) Find the condition of a coefficient \( a(\neq 0) \) that stabilizes the control system represented by the following characteristic equation.

\[
as^3 + 3s^2 + s + 4 = 0
\]

(2) Consider a first-order lag system with the time constant \( T(>0) \).

\[
G(s) = \frac{1}{1 + Ts}
\]

a) Draw the Bode diagram, where the vertical and horizontal axes of the gain diagram should be \( 20 \log_{10} |G(j\omega)| \) and \( \log_{10}(\omega T) \), respectively, and the vertical and horizontal axes of the phase diagram should be \( \arg G(j\omega) \) and \( \log_{10}(\omega T) \), respectively.

b) Find \( \omega \) when the gain is \(-20\)dB.

c) Find the unit step response, and also find the time when the response value reaches within \( \pm 5\% \) from the convergence value. Use \( \ln(0.05) = -3.0 \) if necessary.

(3) Consider a system represented by the following equation. Find the condition of \( a(\neq b) \) so that the indicial response \( y(t) \) has a steady value when a unit step input \( u(t) \) is applied, and also find the steady value. Here \( U(s) \) and \( Y(s) \) are the Laplace transform of \( u(t) \) and \( y(t) \), respectively.

\[
Y(s) = \frac{(s + b)}{(s^2 + s + 2)(s + a)}U(s)
\]

(4) Find the open-loop transfer function of the feedback control system represented by the root locus shown in Fig. 1, where \( K \) is a feedback gain, and the symbol \( \times \) denotes the pole location.
2. Consider a single input system given by the state equation

\[ \frac{dx}{dt} = Ax + bu, \]

where \( x \) is a state variable vector, \( t \) is time, \( A \) is a coefficient matrix, \( b \) is a coefficient vector, and \( u \) is input. Solve the following problems.

(1) Show the definition of the matrix exponential function \( e^A \) of the coefficient matrix \( A \), using \( I \) for the unit matrix.

(2) When the input \( u = 0 \) and the initial state is \( x_0 \), the solution is given by \( x = e^{At}x_0 \). Derive the following relationship,

\[ e^{At} = \mathcal{L}^{-1} \left( (sI - A)^{-1} \right), \]

where \( \mathcal{L}^{-1} \) is the inverse Laplace transformation, and \( s \) is the Laplace operator.

(3) Given the coefficient matrix \( A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \), obtain the transfer matrix \( e^{At} \), and explain about the stability of the system.

(4) Given the input \( u \) as

\[ u = -f^T x, \]

the coefficient matrix as \( A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \), and the coefficient vector as \( b = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \),

obtain the coefficient vector \( f \) so that the poles of the system are \(-3\) and \(-4\).

(5) Given the input \( u \) as

\[ u = -f^T x + u_0(t), \]

where \( f \) is a coefficient vector, and \( u_0(t) \) is the unit step function, obtain the solution of the state equation for the initial state \( x_0 \).
1. A phase diagram of water-air system under atmospheric pressure is given in Fig. 1. The phase
diagram consists of the three phases, liquid water (hereinafter referred to as water), solid water
(hereinafter referred to as ice), and air. Answer the following questions.

(1) Give the names of all the phases existing in the regions (a) to (e) of Fig. 1, respectively.

(2) Water, ice, and air can coexist at only one condition in the phase diagram. Explain the
reason using the Gibb's phase rule.

(3) Water containing 0.002 wt% air was slowly cooled from room temperature. Draw the
schematics of the microstructures just above and below point A in Fig. 1.

(4) In general, when water containing dissolved air freezes, a cloudy ice is grown, because very
fine air bubbles are mixed in the ice. Explain the principle to make a transparent ice as
much as possible from water containing 0.002 wt% air, based on the phase diagram shown
in Fig. 1. De-aeration by pressure reduction or by boiling is excluded.
2. Consider fatigue of metals in an inert environment at room temperature. Answer the following questions.

(1) Explain the reason why the fatigue failures occur due to the application of cyclic stresses that are lower than the yield stress.

(2) Table 1 shows the maximum stress $\sigma_{\text{max}}$ and the minimum stress $\sigma_{\text{min}}$ in the loading cycle of fatigue test. Calculate the mean stress $\sigma_{\text{mean}}$, the cyclic stress amplitude $S$ and the stress ratio $R$ of testing condition I - III, respectively, where tensile stress is positive and compressive stress is negative.

(3) Figure 2 shows a striped pattern which is often observed on fatigue fracture surface.
   a) Give the name of the striped pattern and explain the mechanism of its formation.
   b) Explain the meaning of the pitch of the stripes $d$.

(4) Explain $S$-$N$ curves in fatigue, and describe a difference between $S$-$N$ curves for mild steels and aluminum alloys.

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<th></th>
<th>I</th>
<th>II</th>
<th>III</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{\text{max}}$ (MPa)</td>
<td>240</td>
<td>300</td>
<td>280</td>
</tr>
<tr>
<td>$\sigma_{\text{min}}$ (MPa)</td>
<td>0</td>
<td>60</td>
<td>-280</td>
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</table>

Fig. 2
1. Consider three concentric conducting spherical shells $A$, $B$ and $C$ with thickness of $\delta$ placed in a vacuum as shown in Fig. 1(a). The radii of spherical shells $A$, $B$ and $C$ are $a$, $b$ and $c$ ($a < b < c$), respectively. Both spherical shells $A$ and $C$ are not charged initially, while shell $B$ carries a total electric charge of $Q$ ($Q > 0$). Then, shells $A$ and $C$ are connected by a fine conductive wire passing through a tiny hole on shell $B$. The wire is insulated from shell $B$. Assuming that the thickness $\delta$ and the influences caused by the hole and the wire can be neglected, answer the following questions. Use $\varepsilon_0$ for permittivity.

(1) The electrostatic field in conductors is generally zero. Explain the reason.
(2) Consider a spherical surface $S$ of radius $a$ located in conductor region of shell $A$ as shown in Fig. 1(b). Show Gauss' law in integral form with respect to the surface $S$. Then, find the total charge lying on the inner surface of shell $A$.
(3) The outer surface of shell $A$ carries the charge $Q_A$. Show the charges on the outer and inner surfaces of shells $B$ and $C$, respectively, using $Q_A$ and $Q$.
(4) Show the magnitude and direction of the electric field in vacuum regions inside shell $C$, using $Q_A$ and $Q$.
(5) Show the electric potential at shells $A$, $B$ and $C$, respectively, using $Q_A$ and $Q$.
(6) Find the charge $Q_A$.
2. As shown in Fig. 2(a), a current \( I(t) = I_0 \sin(\omega t) \) flows in a conductive wire placed along the z-axis in the Cartesian coordinate system \((x, y, z)\). Here \( I_0 \) and \( \omega \) are constants. A square coil \( C_1 \) with a side length of \( a \) \((a \ll L)\) is set in the xz-plane and the center of coil \( C_1 \) is located at point \( A_1(L, 0, 0) \). Answer the following questions. Use \( \mu_0 \) for the permeability.

1. Find the magnetic flux density \( B \) induced by the current \( I(t) \) at a point \((x, 0, 0)\), where \( x \neq 0 \).
2. Find the magnetic flux \( \Phi \) through coil \( C_1 \).
3. Find the electromotive force \( \phi^{e.m} \) induced in coil \( C_1 \). Here, the magnetic flux density due to the current flowing in coil \( C_1 \) is negligibly small.
4. Express \( \phi^{e.m} \) obtained in question (3) as a function of \( a/L \). Then, find the first order approximation of \( \phi^{e.m} \) in terms of \( a/L \) on the assumption \( a/L \ll 1 \).
5. Consider the case when the wire is shifted along the x-axis by a small distance \( \Delta L \) \((\Delta L \ll L)\) as shown in Fig. 2(b). Coil \( C_2 \) having the same shape as that of coil \( C_1 \) is added in the xz-plane, and the center of coil \( C_2 \) is located at point \( A_2(-L, 0, 0) \). When the amplitude \( I_0 \) of the current is unknown, express \( \Delta L \) using the electromotive forces \( \phi_1^{e.m}, \phi_2^{e.m} \) induced in coils \( C_1 \) and \( C_2 \).
1. Answer the following questions, where $h$ is Planck's constant.

(1) Assume that an electron of a hydrogen atom is in a circular orbit with the radius $r$ from the hydrogen nucleus, and that the de Broglie wavelength $\lambda_e$ of the electron satisfies $2\pi r = n\lambda_e$ \( (n = 1, 2, 3, \ldots) \). When the hydrogen atom is in the ground state, show the radius $r$ using elemental charge $e$, vacuum permittivity $\varepsilon_0$, electron mass $m_e$, and $h$.

(2) Consider a phenomenon that a photon with wavelength $\lambda$ and energy $E$ collides with a free electron at rest, and is scattered with wavelength $\lambda'$ at the angle $\theta$ with respect to the incident direction. The relationship between $\lambda$ and $\lambda'$ is expressed by $\lambda' - \lambda = \left(\frac{\hbar}{m_0 c}\right)(1 - \cos \theta)$ where $c$ is the velocity of light in vacuum and $m_0$ is the electron rest mass.
   a) Show the relationship between $E$ and $\lambda$.
   b) Show the energy $E'$ of the scattered photon using $E$, $m_0$, $c$ and $\theta$.

(3) When a potential $V(x)$ is an even function in one-dimensional space, wave functions of a particle in $V(x)$ are even or odd functions. Find the normalized wave functions of a particle with mass $m$ in the following potential

$$V(x) = \begin{cases} \infty & (x < -a/2) \\ 0 & (-a/2 \leq x \leq a/2) \\ \infty & (a/2 < x) \end{cases}$$

where $a$ is a positive constant.
2. Answer the following questions, where \( i \) is imaginary unit, \( h \) is Planck's constant, and \( h = h/(2\pi) \).

(1) The \( z \)-component operator \( \hat{\mathbf{L}}_z \) of the orbital angular momentum in three-dimensional space is expressed by \( \hat{\mathbf{L}}_z = -i\hbar \partial/\partial \phi \) in three-dimensional polar coordinates \((r, \theta, \phi)\). Assuming that a function \( \Phi(\phi) \) satisfies \( \hat{\mathbf{L}}_z \Phi(\phi) = m\hbar \Phi(\phi) \), where \( m \) is a real constant.
   a) Find \( \Phi(\phi) \).
   b) When \( \Phi(\phi) = \Phi(\phi + 2\pi) \), find \( m \).
   c) Normalize \( \Phi(\phi) \) in \( 0 \leq \phi \leq 2\pi \).

(2) Assume that the wave function \( \psi(x,t) \) of a particle with momentum \( p \) and energy \( E \) in one-dimensional space simultaneously satisfies the following two equations

\[
-i\hbar \frac{\partial \psi(x,t)}{\partial x} = p \psi(x,t),
\]

\[
 i\hbar \frac{\partial \psi(x,t)}{\partial t} = E \psi(x,t),
\]

where \( x \) is position and \( t \) is time.

a) Find \( \psi(x,t) \).

b) Find the phase velocity of the wave which is expressed by \( \psi(x,t) \).